ETH zürich

Institute for Dynamic Systems and Control

Control Systems 1

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Wrap-Up

The following slides are a very condensed recollection of the things we did, and especially things you could **add onto your summary.**

In my opinion, some of them are **crucial** for you to solve tasks. If you know them by heart, even better. But since you have 40 pages of your own summary, might as well just add in on there.

Also I would suggest **going over the two summaries** I have linked on my polybox, or my website under «Additional Materials», to gain some inspiration for your own one.

There are also a **couple of things missing**. I'm sure you yourself are going to find points that are important to you, so absolutely add them to your summary.

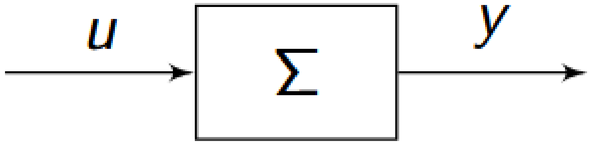
For the exam I would really suggest you **solve a lot of exams** and try to get that **pattern recognition** to be working super fast!

I would greatly appreciate some feedback to the exercise session, so I can gradually improve the structure and style! ☺

jschultev.github.io/personal_website/Feedback

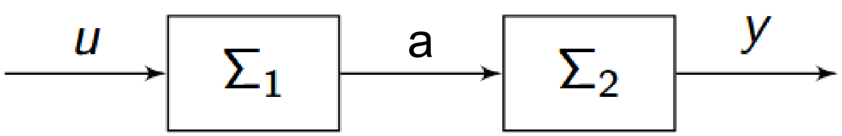
If you dont trust yourself…

Basic



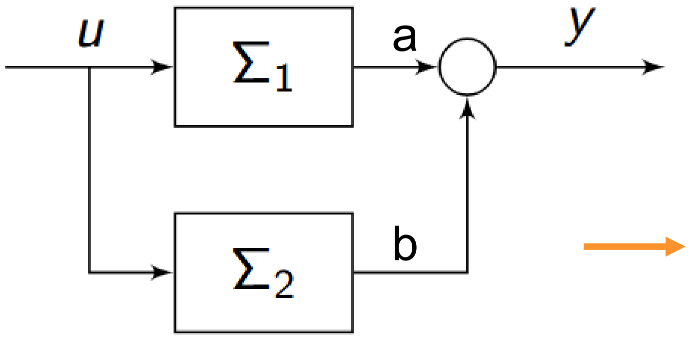
$$y = \Sigma u$$

Serie



$$\left.\begin{aligned} y &= \Sigma_2 a \\ a &= \Sigma_1 u \end{aligned}\right\} \quad y = \Sigma_2 \Sigma_1 u \qquad \Sigma = \Sigma_2 \Sigma_1$$

Parallel



$y = a + b$

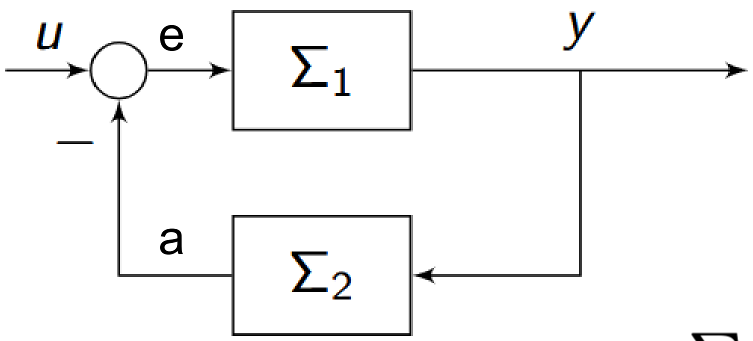
$a = \Sigma_1 u$

$b = \Sigma_2 u$

→ $y = (\Sigma_1 + \Sigma_2) u$

→ $\Sigma = \Sigma_1 + \Sigma_2$

Negative Feedback



→ $\Sigma = \dfrac{\Sigma_1}{(1 + \Sigma_1 \Sigma_2)}$

Linear or Nonlinear

For a System Σ to be linear, we need to have the following properties:

1. Additivity: $\Sigma(u_1 + u_2) = \Sigma u_1 + \Sigma u_2$
2. Homogeneity: $\Sigma(\alpha u) = \alpha \Sigma u \ , \ \alpha \in \mathbb{R}$

Summarized, this leads to the following:

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2, \quad \alpha, \beta \in \mathbb{R}.$$

Examples of ***linear*** systems:

- Integrator: $y(t) = \int_{-\infty}^{t} u(\tau) d\tau$
- Derivative: $y(t) = \dot{u}(t)$
- Time shifts: $y(t) = u(t - \tau)$
- Time scaling: $y(t) = u(t^2)$

Examples of ***non-linear*** systems:

- $y(t) = u(t)^2$
- $y(t) = u(t) + a$
- $y(t) = \sin(u(t))$

Causal or Noncausal

A system is called casual, **iff** (if and only if) the output depends only on past and current inputs, but never on future inputs (future does not change the present).

Only causal systems are physically realizable

Causal systems include:

- $y(t) = u(t)$
- $y(t) = u(t-\tau), \ \forall \tau > 0$
- $y(t) = \cos(3t+1)u(t-1)$
- $y(t) = \int_{-\infty}^{t} u(\tau)d\tau$

Non-Causal systems include:

- $y(t) = u(t-a), \ \forall a < 0$
- $y(t) = u(t+1) + u(t) + u(t-1)$
- $y(t) = u(bt), \ \forall b > 1$
- $y(t) = \int_{-\infty}^{t+1} u(\tau)d\tau$

Static (memoryless) or Dynamic

In static systems, the output only depends on the current input (no past, no future)

Memoryless (or static) systems include:

- $y(t) = 3u(t)$
- $y(t) = t^2 u(t)$
- $y(t) = 2^{-(t+1)} u(t)$
- $y(t) = \sqrt{\sin(u^2(t))}$

Dynamic systems include:

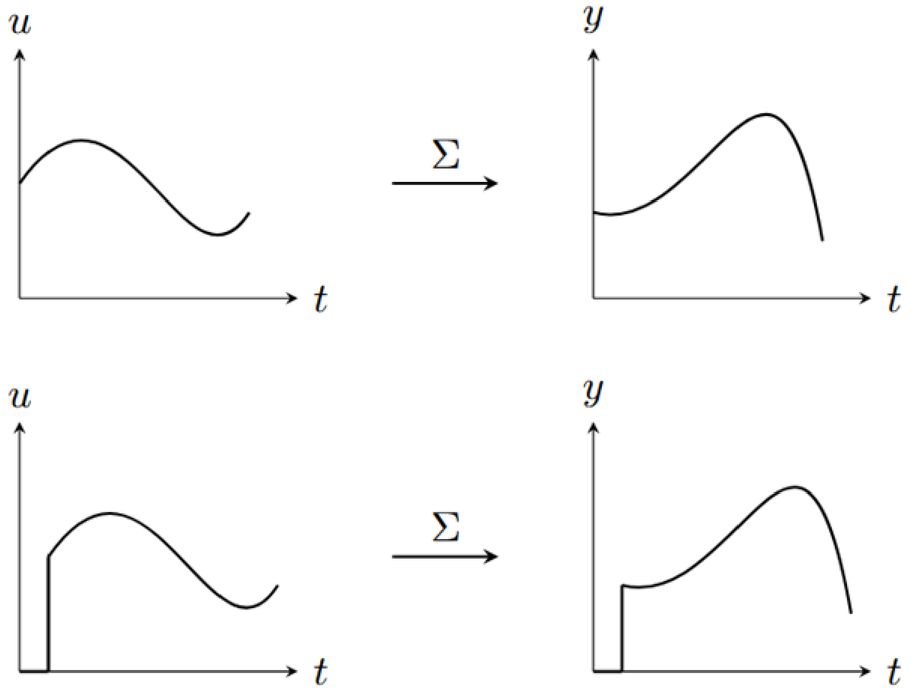
- $y(t) = \int_{-\infty}^{t} u(\tau) d\tau$
- $y(t) = \dot{u}(t)$
- $y(t) = u(t^2)$
- $y(t) = u(t-a), \ \forall a \neq 0$

Time invariant or Time variant

A time invariant system will always have the same output to a certain input, independent of when the input is applied. We can shift the input in time and the output will also be shifted by the same amount.

Example: An ideal lightbulb will always generate the same amount of light, no matter if at night or during the day.

Counterexample: A rocket losing mass (fuel) along it's way and being exposed to less gravitational force. The same thrust (input) leads to a different distance moved (output).



General rule of thumb:The system is time-invariant if the system equations do not contain time t as a summand, factor or exponent. Time t only appears in u(t).

Mathematically: $y(t-\tau) = (\Sigma\tilde{u})(t)$, where $\tilde{u}(t) = u(t-\tau)$.

Time varying systems include:

- $y(t) = \cos(t)u(t)$
- $y(t) = u(3t+1)$
- $y(t) = \sqrt{t-\cos^2(t)}\; u(t^2)$
- $y(t) = \int_{-\infty}^{t-2} e^{-t}u(\tau+2)d\tau$

Time invariant systems include:

- $y(t) = u(t-1)u(t+2)$
- $y(t) = 1 + u(t-1) + 2u(t)$
- $y(t) = 2\int_{-\infty}^{t} u^2(\tau)d\tau$
- $y(t) = \cos(u(t))\int_{-\infty}^{t-2} u(\tau)d\tau$

Specific Example Time-invariancy

Just as an example

$$y(t) = \sqrt{t - \cos^2(t)} \cdot u(t^2)$$

Remember what we want to check: $y(t-\tau) = \Sigma(u(t-\tau))$

$$\textbf{LHS:}\ y(t-\tau) = \sqrt{(t-\tau) - \cos^2(t-\tau)} \cdot u\big((t-\tau)^2\big)$$

$$\begin{aligned}\textbf{RHS:}\ \Sigma u(t-\tau) = \Sigma\tilde{u}(t) &= \sqrt{t - \cos^2(t)} \cdot \tilde{u}(t^2) \\ &= \sqrt{t - \cos^2(t)} \cdot u\big((t-\tau)^2\big)\end{aligned}$$

$$\textbf{LHS} \neq \textbf{RHS} \rightarrow \textbf{Time-variant}$$

State Space Model

LTI SISO

For LTI SISO Systems, we can represent out state space model in the following new form:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t)).$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}u(t).$$
$$y(t) = x_2(t).$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0 \quad u(t).$$

Linearization

For non linear systems to be represented in the state space model, the **general procedure** looks like this:

- Find equilibrium point by solving $f(x_e, u_e) = 0.$
- Linearize around the equilibrium point using Jacobian-Linearization-Procedure, which is basically Taylors-series. More precisely, our matrices will look like this:

$$A = \frac{\partial f(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{(x_e,u_e)}$$

$$B = \frac{\partial f(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}\Bigg|_{(x_e,u_e)}$$

$$C = \frac{\partial h(x,u)}{\partial x}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}\bigg|_{(x_e,u_e)}$$

$$D = \frac{\partial h(x,u)}{\partial u}\bigg|_{(x_e,u_e)} = \begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix}\bigg|_{(x_e,u_e)}$$

Time Response Solution n - Order System

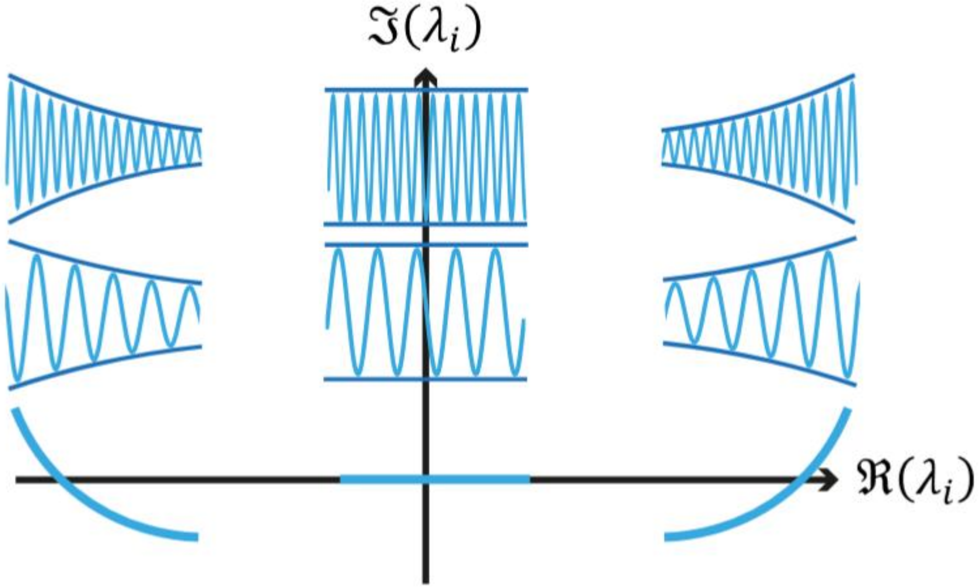
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, \mathrm{d}\tau,$$

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{IC}} + \underbrace{C \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, \mathrm{d}\tau + D\, u(t)}_{\text{F}}.$$

With A, B, C and D being the matrices from the LTI SISO State-Space

$$\boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}}$$

Stability Classification



Asymptotically Stable: State converges to zero for bounded initial conditions and zero input.

$$\mathrm{Re}(s) < 0 \text{ for all } \lambda_i.$$

Lyapunov Stable: State will remain bounded for bounded initial conditions and zero input.

$$\mathrm{Re}(s) \leq 0 \text{ for all } \lambda_i.$$

BIBO Stable: Output remains bounded for every bounded input.

In Linear Systems: **Asymptotically stable → Lyapunov stable**
Asymptotically stable → BIBO stable

Transfer function

$$y(t) = \underbrace{Ce^{At}\left[x_0 - (sI - A)^{-1}B\right]}_{\text{Transient response}} + \underbrace{\left[C(sI - A)^{-1}B + D\right]e^{st}}_{\text{Steady state response}}.$$

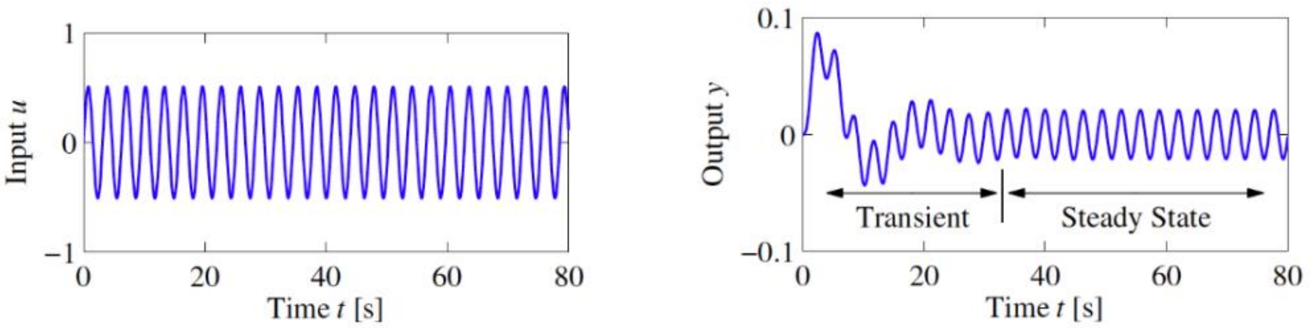
Let us define the **transfer function** G(s) as the following:

$$G(s) = C(sI - A)^{-1}B + D.$$

The steady state response is therefore defined as:

$$y_{ss}(t) = G(s)e^{st}$$

When a system is asymptotically stable, the transient response (with matrix A in exponential) goes to zero! See below:



Transfer Function

In general, the transfer function G(s) (Laplace domain) can be written as:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + D$$

Poles are the roots (Nullstellen) of the denominator

Zeros are the roots of the numerator

There is another form called the **Controllable Canonical Form**. It provides a minimal dimension model to fully describe our system.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & 1 \\ -a_0 & -a_1 & \dots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \end{bmatrix}, \quad D = [d];$$

Transfer Function Notations

$\underbrace{\hphantom{\text{Transfer Function}}}_{\text{Let's call it «TF»}}$

Let's call it «TF»

On the right we have the general form, where we have the ratio of two polynomials + «feedthrough» d:

$$G(s) = \frac{N(s)}{D(s)} = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} + d.$$

There are however some more notations, that can come in quite handy, depending on our goal:

Partial Fraction Expansion

$$G(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} + d,$$

Root Locus Form (next week)

$$G(s) = \frac{k_{rl}}{s^q} \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_{n-q})},$$

Bode Form (in 3 weeks)

$$G(s) = \frac{k_{\text{Bode}}}{s^q} \frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}$$

Magnitdue and Phase Rule sketch

Magnitude

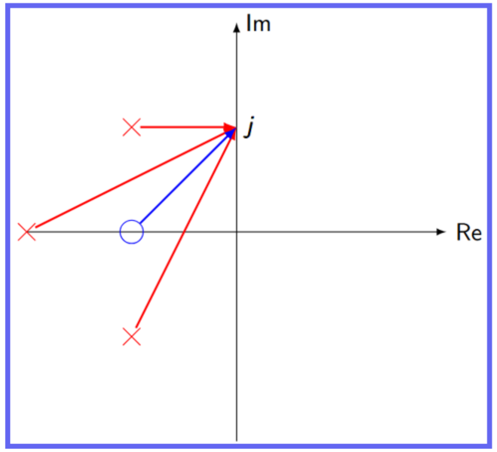
Generally for magnitudes: $|a \cdot b| = |a| \cdot |b|$

Our TF becomes: $|G(s)| = 2\dfrac{|s+1|}{|s+2| \cdot |s+1+j| \cdot |s+1-j|}$

Graphically speaking, |s - p| is just the length of the vector from p to s

Reading from the plot (knowing the position of poles and zeros)

$$|G(j)| = 2\frac{\sqrt{2}}{\sqrt{5} \cdot \sqrt{5} \cdot 1} = \frac{2\sqrt{2}}{5}$$



Phase

Generally for magnitudes: $\angle(a \cdot b) = (\angle a) + (\angle b)$

Our TF becomes:

$$\angle G(s) = \angle(2) + \angle(s+1) - \angle(s+2) - \angle(s+1+j) - \angle(s+1-j)$$

Graphically speaking $\angle(s-p)$ is just the angle formed by vector from p to s with the real axis

Reading from the plot (lowkey):

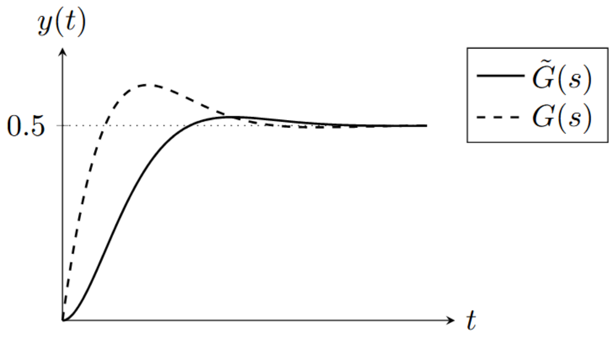
$$\angle G(j) = 0 + 45° - \arctan(1/2) - \arctan(2) - 0° = -45°$$

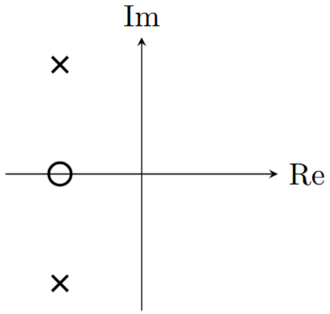
Effect of Zeros

$$\boxed{y(t) = \tilde{y}(t) + \frac{1}{-z}\dot{\tilde{y}}(t)}$$

Minimum Phase Zero

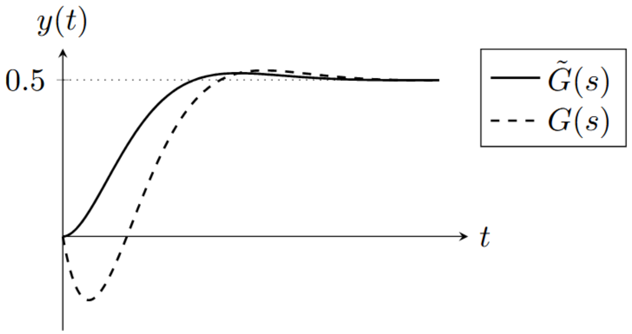
When the real part of the zero lies in the **left half plane**, it is called minimum phase. This adds a **positive** derivative action to the output.

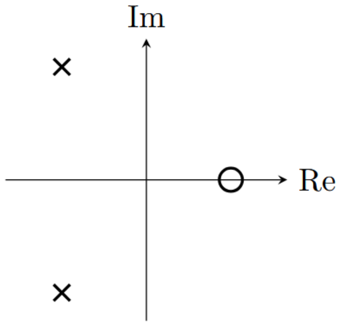




Non Minimum Phase Zero

When the real part of the zero lies in the **right half plane**, it is called non minimum phase. This adds a **negative** derivative action to the output (not good for controls)





Root Locus Rules

1. Root Locus is always **symmetric** with respect **to the real axis**
2. Root Locus always has **as many lines as # of O-L poles** (same degree)
3. Root Locus **starts at O-L poles and ends at O-L zeros** (exception: **Rule 5**)
4. All points on the real axis lie on the root locus. **(sketch from right to left!!!)**
   All points to the left of an **odd** number of zeros + poles are on the **positive** root locus
   All Point to the left of an **even** number of zeros + poles are on the **negative** root locus
5. If there are not enough zeros for every pole, the poles go to infinity along asymptotes

Center of Asympt = $\boxed{S_{COM} = \frac{\sum x_p - \sum x_z}{n-m}}$

n = #Poles
m = #Zeros

$\boxed{\begin{array}{c} \angle(s) = \frac{(2q+1)\cdot 180^\circ}{n-m} \\ q = \{0, 1, ..., n-m-1\} \end{array}}$ = Angle of Asympt.

Some extra rule:

1. Lines enter and leave real axis at 90 degrees

More Root Locus

Rules to follow (a little different than mine, but same)

Asymptotes

Contact point / Centroid of asymptotes

$$\boxed{S_{com} = \frac{\sum x_{Poles} - \sum x_{Zeros}}{\#Poles - \#Zeros}}$$

$$x_i \rightarrow Coordinates\ on\ the\ Real\ axis$$

Angle of asymptotes

$$\boxed{\alpha_n = \frac{2n+1}{\#Poles - \#Zeros} \cdot 180°}$$

$$n = \{0; 1; \ldots; (\#Poles - \#Zeros - 1)\}$$

Sketching Rules

1. Root loci start at poles → go to zeros
2. There are n lines (loci) where n is the degree of Poles or Zeros (whichever is greater).
3. As k increases from 0 to ∞, the roots move from the poles of $G(s)$ to the zeros of $G(s)$.
4. When roots are complex, they occur in conjugate pairs.
5. At no time will the same root cross over its path.
6. The portion (Anteil) of the real axis to the left of an odd number of open loop poles and zeros are part of the loci. → **Roots are always sketched from the right to the left.**
7. Lines leave and enter the real axis at 90°.
8. If there are not enough poles or zeros to make a pair, then the extra lines go to / come from infinity.
9. Lines go to infinity along asymptotes.
10. If there are at least two lines to infinity, then the sum of all roots is constant.
11. K going from 0 to $-\infty$ can be drawn by reversing rule 5 and adding 180° to the asymptote angles.

Cool tool to sketch Root Locus https://lpsa.swarthmore.edu/Root_Locus/RLDraw.html

Bode Plot Rules

1. Write the **TF in Bode form!!**
2. Plot magnitude and phase of **Bode gain**
3. Do not forget to give **magnitude in dB**
4. Superposition all basic compontents. Remember:

   - The **magnitude change** starts **at the position** (frequency) of respective pole or zero
   - The **phase change starts one decade to the left and ends one decade to the right** of the position
   - Integrators and Differentiators have magnitude [dB] = 0 at $\omega = 1$
   - **For complex conjugate** poles remember the **peak and sudden phase change**

1. We are allowed to draw straight lines. Keep in mind however, that this is an approximation.

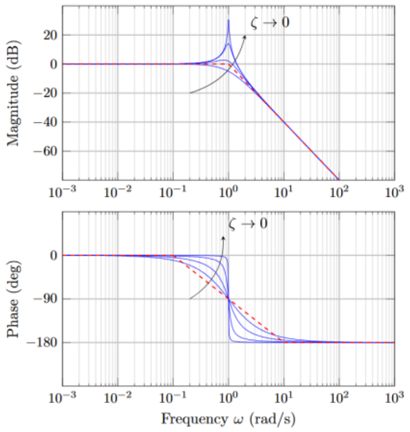
$$\boxed{G(s) = \frac{k_{\text{Bode}}}{s^q}\,\frac{\left(\frac{s}{-z_1}+1\right)\left(\frac{s}{-z_2}+1\right)\cdots\left(\frac{s}{-z_m}+1\right)}{\left(\frac{s}{-p_1}+1\right)\left(\frac{s}{-p_2}+1\right)\cdots\left(\frac{s}{-p_{n-q}}+1\right)}}$$

$$\boxed{|G(j\omega)|[dB] = 20\log_{10}|G(j\omega)|}$$

Term	Magnitude	Phase
Constant K	$20\log_{10}(\lvert K\rvert)$	$\begin{cases} 0^\circ & K>0 \\ \pm 180^\circ & K<0 \end{cases}$
Pole at Origin $\frac{1}{s}$	-20dB/dec	-90° for all ω
Zero at Origin s	$+20$dB/dec	$+90^\circ$ for all ω

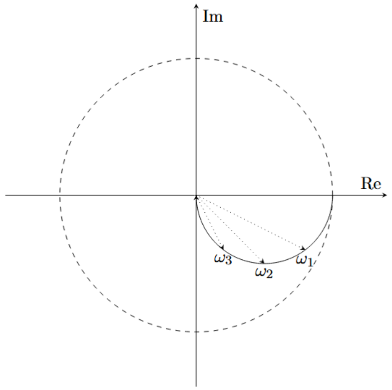
Phase \ Magnitude	-20 dB/dec	$+20$ dB/dec
-90°	stable pole	non-minimum phase zero
$+90^\circ$	unstable pole	minimum phase zero

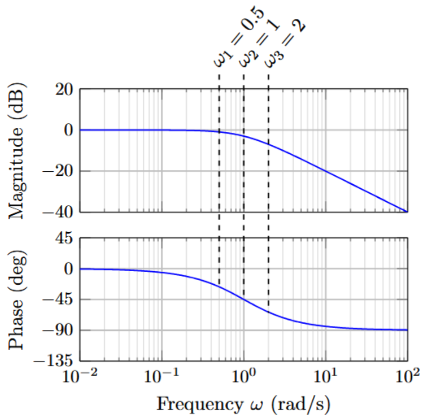
Compl. Conj.



Polar Plot

The Polar plot is a single parametric curve in the complex plane with ω being implicit. Every point on the curve has a certain magnitude and a certain angle (phase), meaning that for every specific ω we get the magnitude and phase of our TF. (Same information as in Bode)





Let $G(s) = \dfrac{N(s)}{D(s)}$ with $\deg N = n$ and $\deg D = m$. As $\omega \to \infty$,

$$G(j\omega) \sim \frac{c_n (j\omega)^n}{a_m (j\omega)^m} = \frac{c_n}{a_m} (j\omega)^{n-m}$$

so

$$|G(j\omega)| \sim \left|\frac{c_n}{a_m}\right| \omega^{n-m} \qquad \angle G(j\omega) \sim (n-m)\cdot 90^\circ + \angle\left(\frac{c_n}{a_m}\right)$$

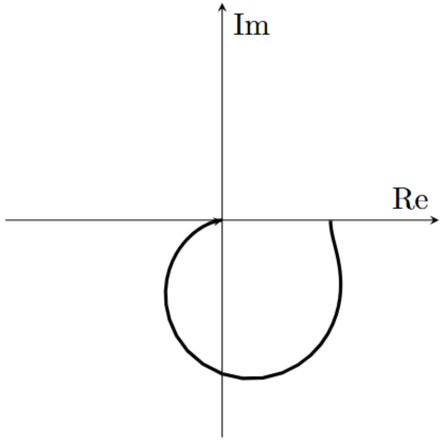
This tells you whether the Polar curve ends at the **origin**, at a **nonzero finite point**, or at **infinity**, and from **which angle** it approaches.

Nyquist Plot Construction

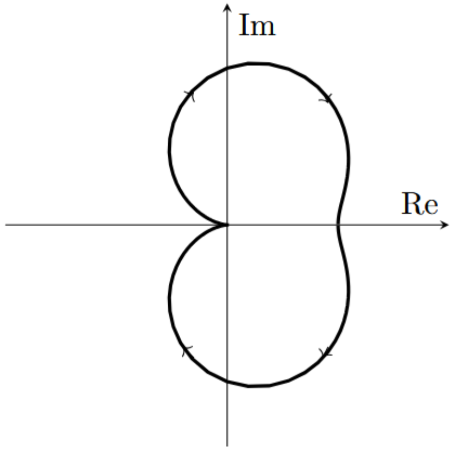
The **Polar plot** is a parametic curve in the complex plane for ω: $0 \to \infty$
The **Nyquist plot** is nothing but the completion of the Idea, meaning ω: $-\infty \to \infty$

Simplistically stated, it is just the Polar plot mirrored on the real axis, but continuing the direction of the parametric curve.

Polar Plot



Nyquist Plot



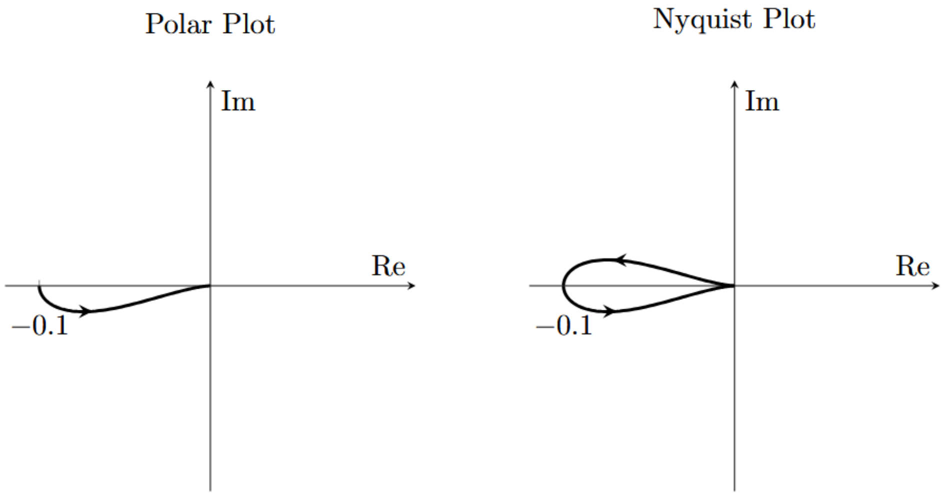


Figure 6.32: Polar and Nyquist plots of $L(s) = \frac{1}{s^2+s-10}$.

Nyquist

With Nyquist, we will always be able to assess the closed loop stability, always!
No matter how the open loop system looks like

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$

$$Z = N + P$$

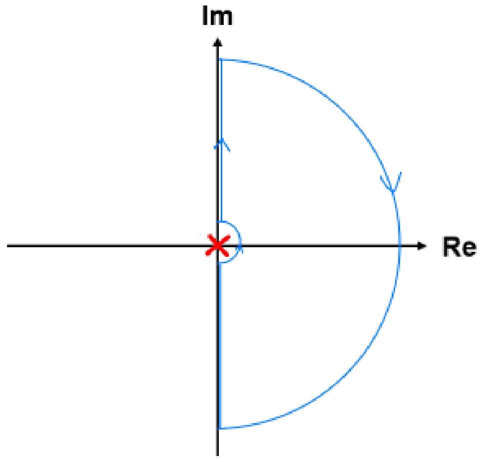
Z = # Unstable Closed-Loop Poles
N = # CW enc. of $L(s)$
P = # Unstable Open-Loop Poles

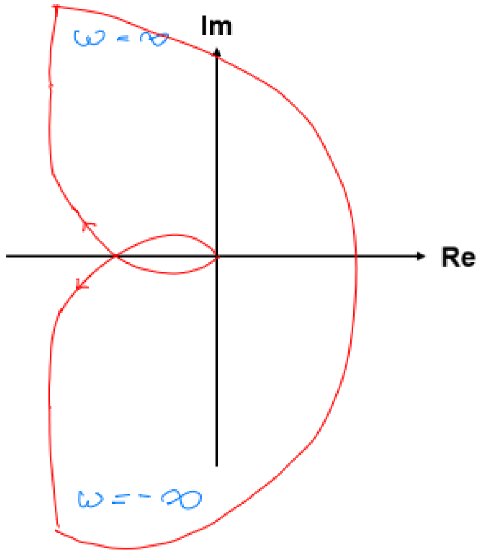
This is awesome because it allows us to assess the closed loop stability by only looking at the open loop system $L(s)$

Imaginary Poles with Nyquist

What happens when there are poles on the Im – axis of the open loop system? Do they count into the Contour or not?
For that we will just follow a special rule:

When excluding the poles, going around them CCW, close the Nyquist plot at infinity clockwise.
For every pole excluded, add $+180^\circ$ when closing the Nyquist

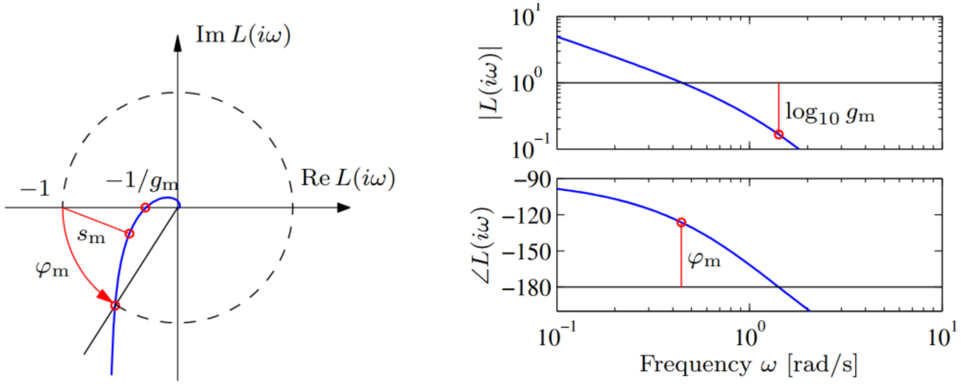




Stability Margins

Gain Margin GM / $g_{m,dB}$: It indicates how much we can increase the magnitude at the phase crossover frequency ω_{pc}, at which the phase crosses $\pm 180°$, before encircling -1

Phase Margin PM / φ_m: It indicates much much we can change the phase at the crossover frequency ω_c, at which the magnitude is 1 (0 in dB), before encircling -1

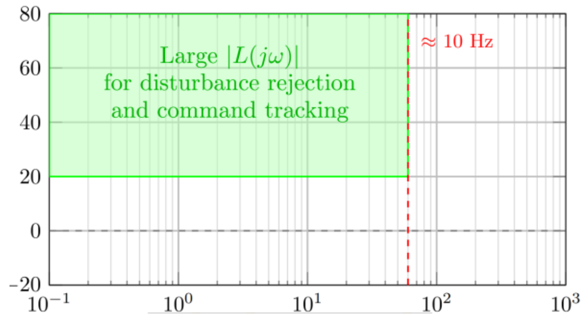


Phase margin:	Gain margin:
Find **cross-over frequency** ω_c:	Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $\lvert L(j\omega)\rvert_{dB}$ for the same ω
$\lvert L(j\omega_c)\rvert = 1$ $\lvert L(j\omega_c)\rvert_{dB} = 0$	Compute the gain margin:
Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin:	$g_{m,dB} = 0 - \lvert L(j\omega)\rvert_{dB}$ $g_m = 10^{\frac{g_{m,dB}}{20}}$
$\varphi_m = \varphi + 180°$	

Frequency Domain Specifications

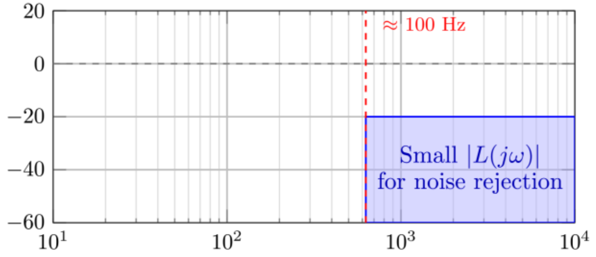
Since $S(s)$ maps the **disturbance signal $d(t)$ to the output $y(t)$,** we want $S(s)$ to be small for frequencies in which disturbances are present, therefore for low frequencies.

$$\boxed{|S(j\omega)| \ll 1 \quad \text{for low } \omega \qquad |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \quad \Rightarrow \quad |L(j\omega)| \gg 1}$$



Since T(s) maps the **noise signal n(t) to the output $y(t)$,** we want T(s) to be small for frequencies in which noise is present, therefore for high frequencies.

$$\boxed{|T(j\omega)| \ll 1 \quad \text{for high } \omega \qquad |T(s)| = \left|\frac{L(s)}{1+L(s)}\right| \quad \Rightarrow \quad |L(j\omega)| \ll 1}$$



$$\boxed{|T(s)| \approx |L(s)|}$$

Bandwidth

Hint: Often the Bandwidth is asked in Hz, then use $f = \frac{\omega}{2\pi}$

The **Bandwidth (BW)** of a system tells us the maximum frequency for which our system can track a command with a factor of ≈ 0.7, which corresponds to

$$|T(j\omega)| > \frac{1}{\sqrt{2}} \quad (\approx -3\mathrm{dB})$$

It can be interpreted as **how fast** the closed-loop can react to changing references.

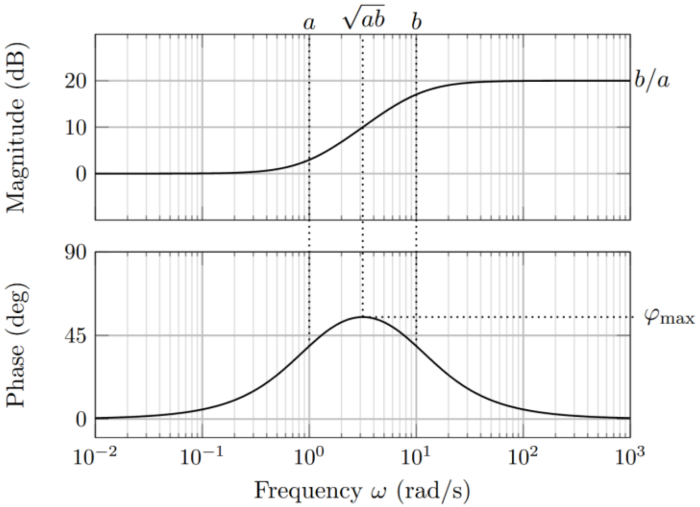
Through some derivation we find, that $\omega_{\mathrm{bw}} \approx \omega_c$
Meaning, our open loop crossover frequency is equal to our closed loop bandwidth

For this assumption to hold, we will always want a slope of $-20\mathrm{db/dec}$ at the crossover frequency

Lead Compensator

$$C_{\text{lead}}(s) = \frac{s/a+1}{s/b+1}, \quad 0 < a < b$$

Minimum Phase Zero before Stable Pole



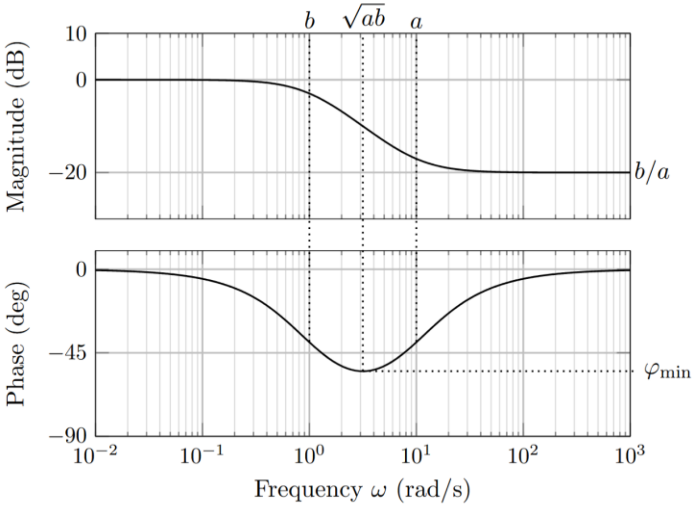
- Magnitude increase of $\frac{b}{a}$
- $+20\text{dB/dec}$ between a and b
- Maximum phase **increase**

$$\varphi_{\max} = 2\arctan\left(\sqrt{\frac{b}{a}}\right) - 90°$$

Lag Compensator

$$C_{\text{lag}}(s) = \frac{s/a+1}{s/b+1}, \quad 0 < b < a$$

Stable Pole before Minimum Phase Zero



- Magnitude decrease of $\frac{b}{a}$
- -20dB/dec between a and b
- Maximum phase **decrease**

From / For ZF

Lead Compensators

A lead compensator is a loop element that increases the phase margin. However, it also makes a system more sensitive to noise.

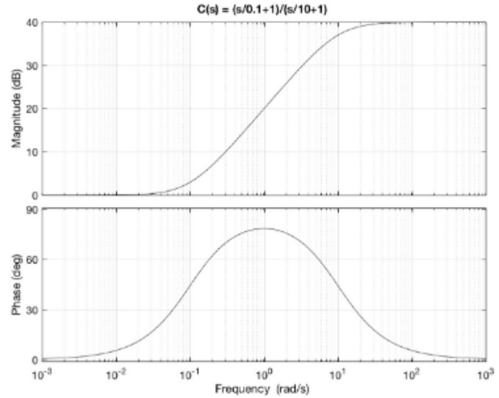
$$\boxed{C_{\text{lag}} = \frac{s/a+1}{s/b+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}, \quad 0<a<b}$$

$|L(j\omega)|$ changes by $+20$dB/dec between a and b. The maximum increase of φ is:

$$\boxed{\Delta\varphi \approx 2\cdot\arctan(\sqrt{b/a}) - 90^\circ}$$

Design procedure:

1. Pick the desired crossover frequency $\omega_c = \sqrt{ab}$
2. Pick b/a depending on the desired phase increase
3. Possibly add a proportional gain k to set ω_c back to the desired frequency ($|kL(jw)| = 1$)



Lag Compensators

A lag compensator is a loop element that improves disturbance rejection and command tracking by decreasing the sensitivity to noise. However, it also decreases the phase margin of the system and can hence make it unstable:

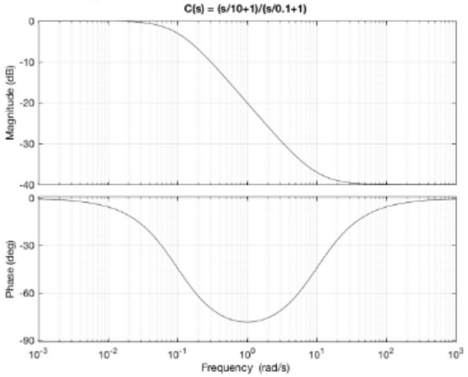
$$\boxed{C_{\text{lag}} = \frac{s/a+1}{s/b+1} = \frac{b}{a}\cdot\frac{s+a}{s+b}, \quad 0<b<a}$$

$|L(j\omega)|$ changes by -20dB/dec between a and b. The maximum decrease of φ is:

$$\boxed{\Delta\varphi \approx 2\cdot\arctan(\sqrt{b/a}) - 90^\circ}$$

Careful: As the phase decreases, $\Delta\varphi$ is negative.
Design procedure:

1. Choose a/b as the desired increase in magnitude at low ω.
2. Pick $\sqrt{ab}$ as far as possible from the desired crossover frequency ω_c to not risk instability.



Time Delay

Real systems have a finite computation time and / or transmission time, meaning the control input has some delay compared to the reference signal.

Time Delays will shift the input signal $u(t)$ by $T \geq 0$ units of time.

We find the TF of the time delay operator by $\boldsymbol{e^{-sT}}$
The Time-Delay operator is **Linear** and **Time-Invariant** so **(LTI),** but unfortunately it is **not rational**.

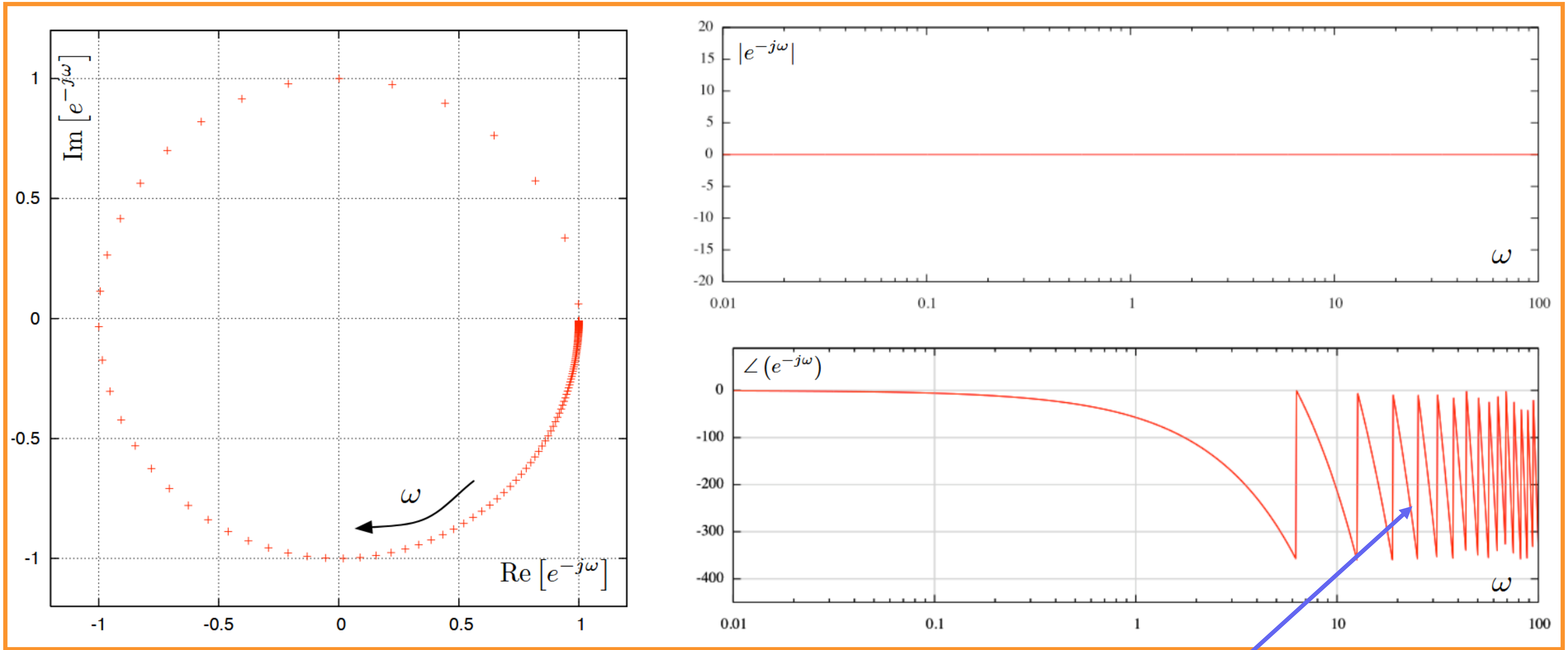
Let's look at the frequency response. Plugging in $s = j\omega$ we get:

$$\boxed{\left|e^{-j\omega T}\right| = 1} \qquad \boxed{\angle\left(e^{-j\omega T}\right) = -\omega T}$$

From this we can deduce 2 important consequences:

- The **Magnitude is constant** and equal to one for all frequencies
- The **phase decreases linearly** with ω, meaning:a sinusoidal signal gets more and more shifted

Time Delay Plot



Phase going to $-\infty$

Time Delay Consequences

$$\boxed{L'(j\omega) = e^{-j\omega T} L(j\omega)}$$

Crossover frequency

We will now ompare L' with L.

Looking at the phase margin, for the system without time delay we find $\varphi_{m,0} = 180^\circ + \angle L(j\omega_c)$

For the time delay system:

$$\boxed{\varphi_{m,T} = \varphi_{m,0} - \omega_c T}$$

- For a fixed delay T, **increasing the crossover frequency ω_c reduces the phase margin.**
- For a fixed crossover frequency, **increasing the delay T reduces the phase margin**.

Padé Approximation

As already mentioned, e^{-sT} is not a rational function. But we would still like to do root locus! Therefore, we will approximate this exponential time delay as a ratio of two polynomials (a.k.a. Padé approximation)

$$e^{-sT} \approx k\,\frac{s+p}{s+q}$$

To get the coefficients we will compare it to the taylor series expansion: $k\frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2$

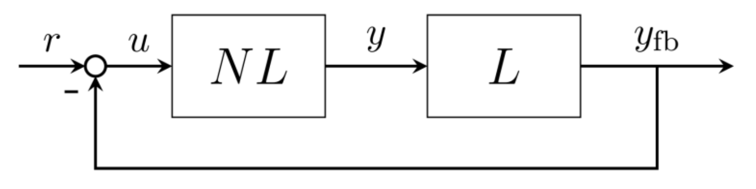
Finally we find:

$$\boxed{e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}}$$

Which means that we can now handle the our TF just as at the beginning, using tools like root locus.

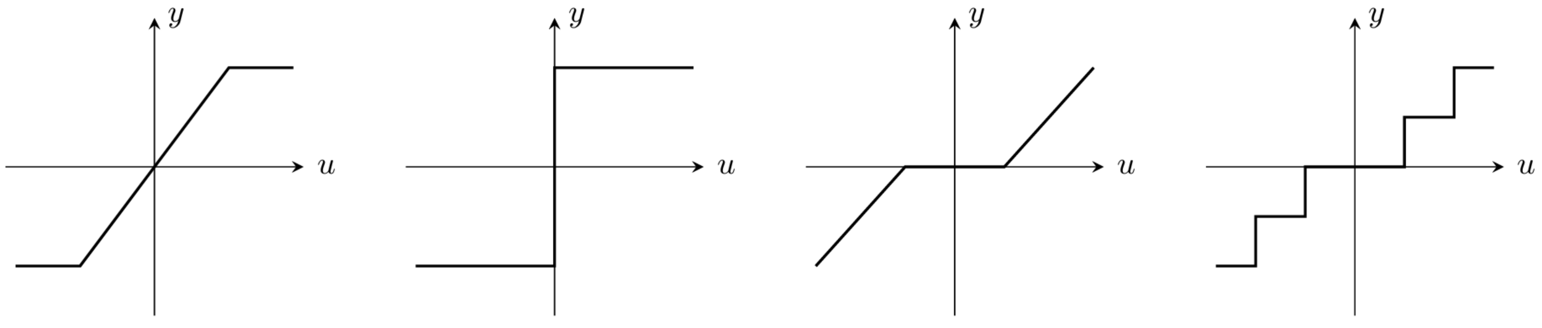
NonLinearities

Typical nonlinear systems we will consider look like the one below. **$L(s)$ is still the linear open-loop TF**, but $\boldsymbol{NL}$ now being the **NonLinearity**. The $\boldsymbol{y_{\text{fb}}}$ is then the actual feedback signal.



But what even are those **NonLinearities (NL)**?

Example: Imagine your Innoprojekt motors for the weels of the robot.
They cannot go infinitely fast. They have a certain torque limit they can provide and anything over that value just cannot get realised. This behaviour would correspond to plot a), the **saturation.**



(a) Saturation **(b)** Switch or Relay **(c)** Dead Zone **(d)** Quantizer

Static, memoryless NonLinearities

Absolute Stability

Just as for linear systems, we would also like to assess the stability of nonlinear systems.

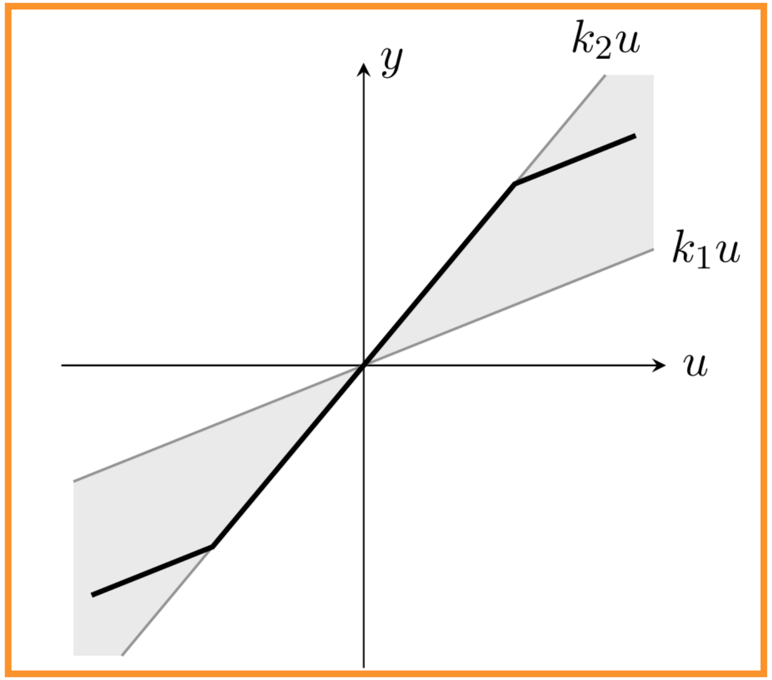
A common and good approach, defining a boundaries that include all values of NL.

$$NL(0) = 0 \quad uk_1 \leq NL(u) \leq uk_2 \quad \forall u \neq 0$$

The closed-loop is said to be absolutely stable, if u = 0 is a globally asymptotically stable equilibrium of the closed-loop system.

Mathematically: for any initial condition of $L(s)$

$$\lim_{t\to\infty} u(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0$$



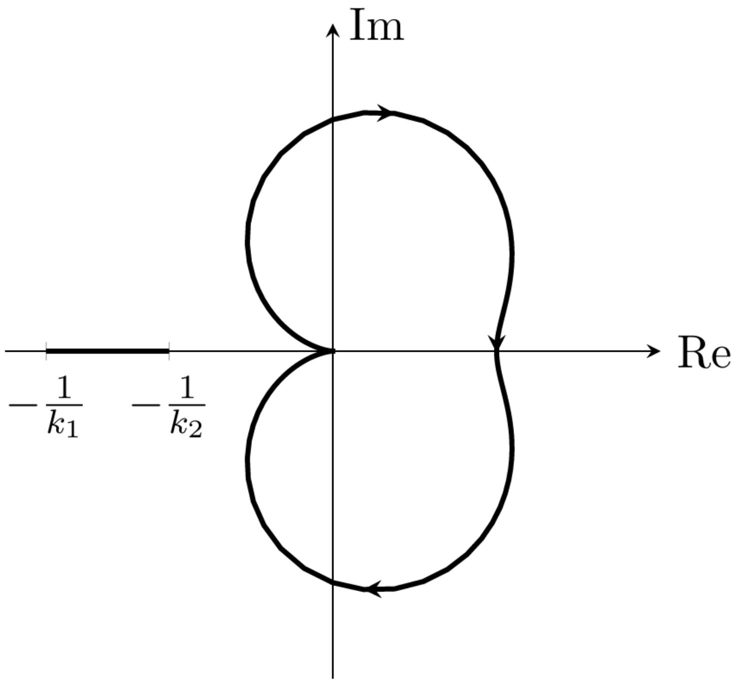
Not super important

Necessary Condition

The way we defined **NL**, it can include all gains $k_1 \leq k \leq k_2$

Now looking at the Niqyuist criterion (at which we look because we are interested in stability), we see that we now not only have to consider $-\frac{1}{k}$, but in the entire interval $[-\frac{1}{k_1}, -\frac{1}{k_2}]$

This means that we actually have to count the correct number of encirclements for the segment $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ in order to **assess stability**

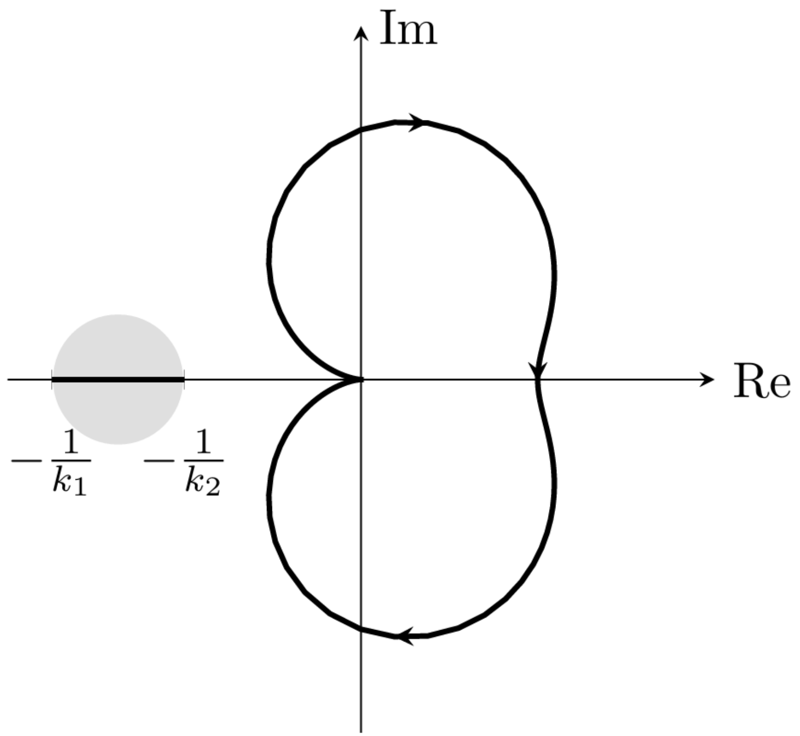


Sufficient Condition, a.k.a Circle Criterion

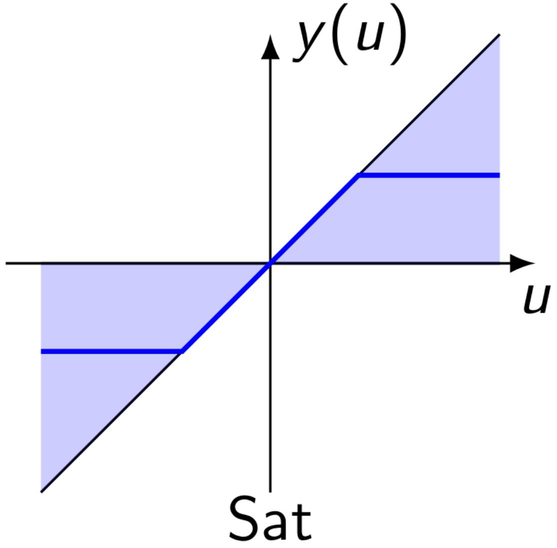
We can be even more careful and define a larger region we have to avoid.
This will lead us to a **circular region** with diameter $[-\frac{1}{k_1}, -\frac{1}{k_2}]$ that we will have to avoid.

This condition is **sufficient**, meaning that if it is fullfilled, we are in fact stable

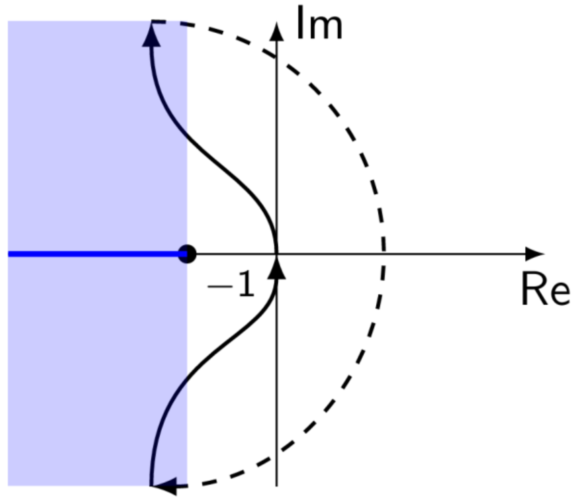
It is however **not necessary** (in contrary to the condition before), meaning that if the condition is not fullfilled, **we cannot make any statement about stability!**



Example



We have a NL with the sector $[\mathbf{0}, \mathbf{1}]$



The segment and circle we may not touch / encircle is therefore $[-\infty, -\mathbf{1}]$

It can be seen, that we touch this region (a circle with infinte diameter, looking like a square).

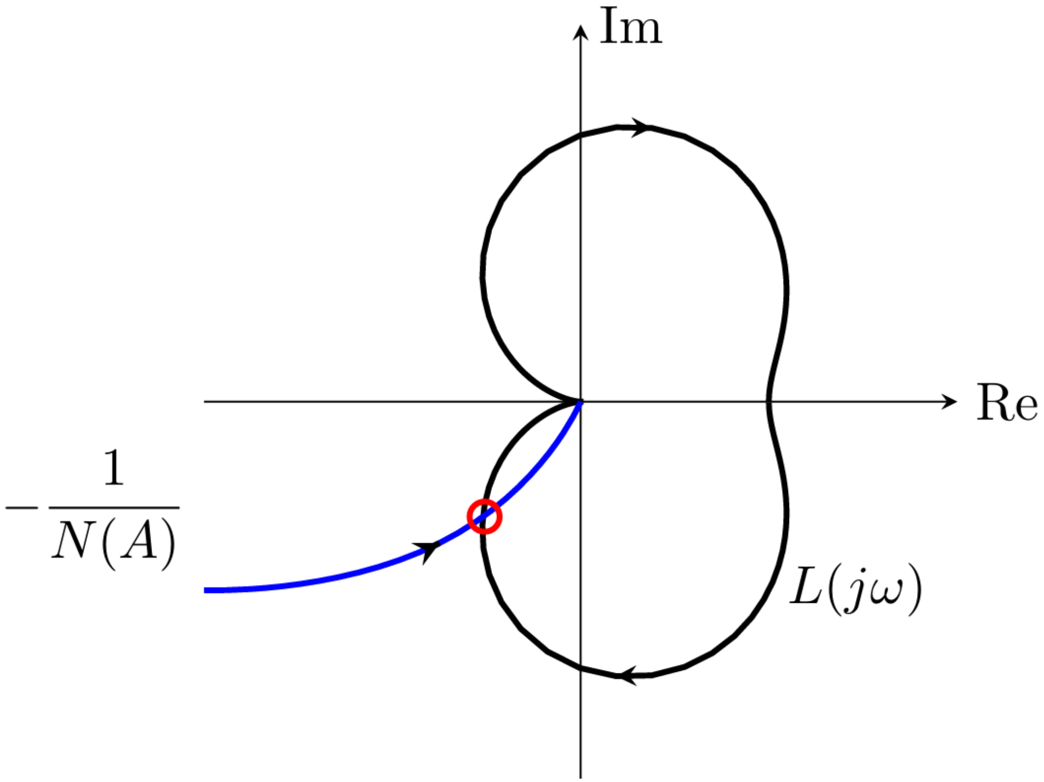
Since this **only violates the circle criterion** and not the necessary condition, we **cannot make any statement about stability.**

Checking for Limit Cycles

When plotting $L(j\omega)$ and $-\frac{1}{N(A)}$, finding intersection points corresponds to finding limit cycles.
$N(A)$ is the describing function, an **approximation of the NL as an amplitude-dependent gain**

Finding intersections fullfillfs the equation below, which was used for the derivation of limit cycles.

$$\boxed{-\frac{1}{N(A)} = L(j\omega)}$$



Limit Cycle Stability

Limit Cycles can be

- **Stable**: The system converges to them, and small deviations from the exact amplitude dont matter
- **Unstable**: For small deviations we go away from this limit cycle

To the right is the nyquist plot of the OL-stable $L(j\omega)$, and the plot of $-\frac{1}{N(A)}$. The arrow indicates in which direction the function goes, if A is increased!

You can also see some regions marked.

- **S** stands for **stable**
- **U** stands for **unstable**

If the point $-\frac{1}{N(A)}$ (this is our new $-\frac{1}{k}$) is in:

- **Unstable region:** The amplitude **A** tends to **increase.**
- **Stable region**: The amplitude **A** tends to **decrease**